

Conservative Rezoning Algorithm for Generalized Two-Dimensional Meshes*

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A method is presented for transferring a conserved quantity Q from one generalized mesh to another when the volumetric density of Q is uniform within each cell of the original mesh.

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Multidimensional Lagrangian hydrodynamics calculations must often be terminated prematurely because of severe distortions in the computing mesh. To continue the calculation, it is then necessary to transfer the conserved quantities (such as mass, momentum, and energy) from the original distorted mesh to a new regular mesh. This process is called rezoning. A rezoning method for arbitrary quadrilateral meshes in two dimensions has recently been described by Dukowicz [1]. Although this method can accommodate higher-order interpolations, the most important case in practice is that in which the volumetric densities of the conserved quantities are considered uniform within each cell of the original mesh (and ultimately the new mesh as well). In this special case, the rezoning problem reduces to the calculation of the volumes of overlap between cells of the original and new meshes. Our purpose here is to point out that this calculation can be performed in a particularly straightforward way. The procedure for doing so is similar in spirit to (and was in fact suggested by) the method of Dukowicz, but is simpler and more direct. It applies equally well in Cartesian and cylindrical coordinates, and formulas for both cases will be given.

By far the most common type of generalized mesh is the arbitrary quadrilateral mesh, which is convenient to work with because it has the same simple topological and logical structure as a square or rectangular mesh. The arbitrary quadrilateral mesh is accordingly adopted as a framework for the discussion. As will be seen, however, the basic procedure is remarkably indifferent to the topology of the mesh

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and is therefore likely to be useful for two-dimensional triangular meshes (and perhaps other meshes) as well.

Restriction to the case in which densities are uniform within cells is tantamount to calculating the associated transport by upwind or donor cell differencing [1]. This type of differencing is quite diffusive, and the present method is therefore not intended or recommended for rezoning on every time step, as is done in Eulerian or certain mixed Lagrangian–Eulerian calculations. It is expected that the method will be useful primarily for the occasional rezoning necessitated by mesh distortion during Lagrangian calculations.

We first treat the case of Cartesian coordinates in the plane, denoted as usual by (x, y) . Our basic building block is the formula for the area of an arbitrary polygon P . Let the sides of the polygon be labeled by the index s . The coordinates of the two endpoints of side s will be denoted by (x_1^s, y_1^s) and (x_2^s, y_2^s) . Which of these endpoints is considered point 1 is immaterial. (However, when side s is a segment of a mesh line, as it will be in what follows, it is usually convenient to let point 1 be the endpoint with the lesser value of the spatial index which varies along the line.) It is convenient to regard each side s as a *directed* line segment from point 1 to point 2, so that one may unambiguously refer to areas lying to the left and right of side s . These terms are defined with reference to an observer standing on the plane at point 1 and facing point 2. The area of the polygon P is then given by

$$A_P = \frac{1}{2} \sum_s \varepsilon_s^P (x_1^s y_2^s - x_2^s y_1^s), \quad (1)$$

where the summation is over all the sides of P , and ε_s^P is either $+1$ or -1 according as P lies to the left or right, respectively, of side s . Note that the endpoint coordinates (x_1^s, y_1^s) and (x_2^s, y_2^s) are considered to be associated with the side s as a separate entity, and not with the particular polygon P per se. These coordinates, and the orientation of side s , would therefore remain the same for a second polygon P' to which side s is common. In contrast, ε_s^P is associated with both side s and polygon P , as the notation indicates, and $\varepsilon_s^{P'}$ would clearly be the negative of ε_s^P . (If P lies to the left of side s then P' would lie to the right, and vice versa.) Equation (1) is a well-known formula [2] which is easily derived by integrating the identity $\nabla \cdot \mathbf{r} = 2$ (where $\mathbf{r} = xi + yj$ is the position vector) over the area of P and applying the divergence theorem.

Now consider a two-dimensional region which is subdivided into arbitrary quadrilateral cells by the lines of the old (original) mesh, and in a different way by the lines of the new mesh. The superposition of both meshes defines a network of elementary overlap areas which are not further subdivided by lines of either mesh (see Fig. 1). Each such overlap area lies entirely within a single cell of the old mesh and a single cell of the new mesh. The overlap areas are polygons whose sides are segments of the old-mesh lines and the new-mesh lines. The number of sides of each type, and the total number of sides, will be different for different overlap areas. Each side is common to two overlap areas, the one on the left (L) and the one on

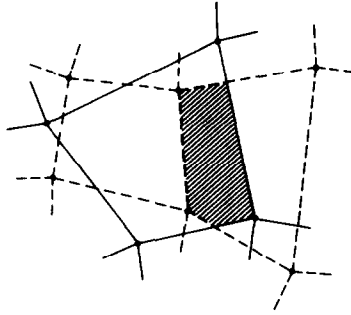


FIG. 1. Overlap areas arising from the superposition of the old mesh (solid lines) and the new mesh (dashed lines). A typical elementary overlap area is shaded.

the right (R), and these overlap areas may be considered to be associated with the side. A side which is an old-mesh segment is also common to two cells of the old mesh, and a side which is a new-mesh segment is also common to two cells of the new mesh. In either case, these cells may be unambiguously referred to as the L and R cells associated with the segment in question.

Our objective is to apportion a conserved quantity Q , whose volumetric density q (Q per unit volume) is considered uniform within each cell of the old mesh, into the cells of the new mesh. (For example, if Q represents the mass M then q is the mass density ρ .) Consider a particular overlap area, and let q be the density of the old-mesh cell in which it lies. The overlap area A is given by Eq. (1), and the quantity $\Delta Q = qA$ is the contribution of this particular overlap area to the total Q of the new-mesh cell in which it lies. (We assume unit depth, so that areas and volumes have the same numerical values.) The final value of Q in this new-mesh cell is then the sum of the ΔQ contributions from all overlap areas that lie within it, and division by the cell area then yields the final value of the density q . It is clear that there is no creation or destruction of the quantity Q at any stage of the procedure, so the rezoning is conservative.

It would be impractical, however, to actually perform the rezoning calculation in the manner just outlined. The key to making the procedure practical is the realization that according to Eq. (1), each of the overlap increments ΔQ is itself the sum of contributions associated with the individual sides of the overlap area. Thus, instead of naively sweeping over the overlap areas directly (which would require complex logic that would be inefficient and difficult to automate in a computer program), one evaluates the same contributions in a different order by simply sweeping over the sides or segments s . Each side s is common to two overlap areas and hence contributes to two ΔQ 's, namely ΔQ_L and ΔQ_R . Since both of these contributions involve the common factor $\frac{1}{2}(x_1^s y_2^s - x_2^s y_1^s)$, it is convenient to compute them at the same time. Of course, the way in which these contributions are calculated and used depends upon whether side s is a segment of the old mesh or the new mesh.

If side s is a segment of the old mesh then it is common to two adjacent cells of the old mesh, the L and R cells. The L overlap area associated with side s lies entirely within the L cell, whose density is q_L , while the R overlap area lies entirely within the R cell, whose density is q_R . Both overlap areas lie entirely within the same cell of the new mesh, and therefore the contributions of side s to ΔQ_L and ΔQ_R may be added to obtain the corresponding increment to Q in this cell. According to Eq. (1), the contribution of side s to ΔQ_L is $\frac{1}{2}q_L(x_1^s y_2^s - x_2^s y_1^s)$, while that to ΔQ_R is $-\frac{1}{2}q_R(x_1^s y_2^s - x_2^s y_1^s)$. The quantity Q in the new-mesh cell containing side s is therefore to be incremented by an amount

$$\Delta_s^0 = \frac{1}{2}(q_L - q_R)(x_1^s y_2^s - x_2^s y_1^s). \quad (2)$$

If side s is a segment of the new mesh then it is common to two adjacent cells of the new mesh, the L and R cells. The L overlap area associated with side s lies entirely within the L cell, while the R overlap area lies entirely within the R cell. Both overlap areas common to side s lie entirely within the same cell of the old mesh, whose density will be denoted by q_0 . The contribution of side s to ΔQ_L will then be an increment to Q_L , while that to ΔQ_R will be an increment to Q_R . The former contribution is

$$\Delta_s^N = \frac{1}{2}q_0(x_1^s y_2^s - x_2^s y_1^s), \quad (3)$$

and the latter is just $-\Delta_s^N$. Therefore Q_L is to be incremented by Δ_s^N , while Q_R is to be decremented by the same amount.

When a new-mesh segment coincides with an old-mesh segment, ambiguities arise and supplementary rules are needed to deal with them. The old-mesh segment does not then lie entirely within a single new-mesh cell but is common to two such cells, so that the new-mesh cell to which the increment Δ_s^0 is to be given is ambiguous. Similarly, the new-mesh segment does not lie entirely within a single old-mesh cell but is common to two such cells, and the density q_0 to be used in Δ_s^N is also ambiguous. One readily verifies that a proper treatment of such cases results if the ambiguities are resolved in a symmetrical way by adoption of the following rules. The increment Δ_s^0 for the old-mesh segment is evaluated in the usual way, and then Q in each of the two common new-mesh cells is incremented by $\frac{1}{2}\Delta_s^0$. The density q_0 for the coincident new-mesh segment is taken to be the average of the densities in the two common old-mesh cells, and then Δ_s^N is evaluated and used in the usual way. Of course, it is necessary to ensure that segment coincidences are correctly identified as such on *both* mesh sweeps, and this requires some care because of roundoff errors. An alternative approach, which is rather pedestrian but fully effective, is to simply destroy any such coincidences by very slightly displacing the vertices of the new mesh in regions where coincidences might otherwise occur.

It is convenient to sweep over all sides or segments s by sweeping along the lines of the old and new meshes, keeping track at all times of which cell of the other mesh the current segment is in. The required logic for doing so, including the associated testing for intersections that determine segment endpoints, is easily

automated and has exactly the same structure as that used to compute line integrals over the new-mesh lines in the Dukowicz method [1] (see Appendix). Here, however, we must sweep over the old-mesh lines as well as the new-mesh lines, but the same logical structure can handle both tasks simply by interchanging the roles of the two meshes. It is noteworthy that this logical structure is the only aspect of the procedure which depends on the topological character of the mesh. The basic procedure is therefore not inherently limited to arbitrary quadrilateral meshes. A corresponding logical structure for two-dimensional triangular meshes is readily envisioned, and analogous logic for other types of meshes could no doubt also be devised.

The general procedure may therefore be summarized as follows. One first sweeps through all the lines of the old mesh, evaluating A_s^O for each segment thereof by means of Eq. (2). The value of Q in the new-mesh cell containing segment s is then incremented by A_s^O . Next one sweeps through all the lines of the new mesh, evaluating A_s^N for each segment thereof by means of Eq. (3). The value of Q in the new-mesh cell to the left of segment s is then incremented by A_s^N , and the value of Q in the new-mesh cell to the right is decremented by A_s^N . Ambiguities arising from coincident old- and new-mesh segments are to be resolved symmetrically, as discussed above. Finally, after both mesh sweeps are complete, the resulting values of Q in the new-mesh cells are divided by the corresponding new-mesh cell areas to obtain the new-mesh cell densities q . The procedure involves one sweep through each mesh, as does the Dukowicz method. However, the present procedure is simpler, as it does not require computation and storage of auxiliary cell constants. It is conceptually simpler as well, provided of course that one is concerned only with the case in which densities are considered uniform within each cell.

Extension of the procedure to cylindrical coordinates (r, z) is straightforward. The volume of revolution of a polygon P about the coordinate axis is given in terms of the side endpoint coordinates (r_1^s, z_1^s) and (r_2^s, z_2^s) by

$$V_P = \frac{\pi}{3} \sum_s \epsilon_s^P (r_1^s + r_2^s)(r_1^s z_2^s - r_2^s z_1^s). \tag{4}$$

This formula does not appear to be as well known as Eq. (1), but it may be easily derived in a similar way. Comparison with Eq. (1) shows that to treat the cylindrical case, it is merely necessary to replace $\frac{1}{2}(x_1^s y_2^s - x_2^s y_1^s)$ by $(\pi/3)(r_1^s + r_2^s)(r_1^s z_2^s - r_2^s z_1^s)$ in the expressions developed for the Cartesian case. This must be done, in particular, in Eqs. (2) and (3). Once this replacement has been made, the entire procedure as already described applies equally to the cylindrical case (except of course that one now divides Q by volume rather than area to obtain the density q).

Finally, it may be of interest to observe that the present rezoning method can be interpreted as a special case of an extended Dukowicz method in which the Dukowicz flux continuity condition (Eq. (5) of Ref. [1]) is relaxed. In the Cartesian case, such an interpretation can be achieved by defining a flux vector $\mathbf{F} = \frac{1}{2}q\mathbf{r}$, where

q is the piecewise uniform density field defined by the old mesh. This flux satisfies the condition $\nabla \cdot \mathbf{F} = q$ in the *interior* of every cell in the old mesh, but $\nabla \cdot \mathbf{F}$ now contains delta functions at the old cell boundaries which are not present in the Dukowicz method. Consequently, the integral of $\nabla \cdot \mathbf{F}$ over an arbitrary cell of the new mesh is no longer simply the value of Q in that cell. This complication can be dealt with, however; the amplitudes of the delta functions are simply related to the discontinuities in q at the old cell boundaries, and their contributions can be explicitly evaluated. When this is done, the net result is precisely the present method. Of course, this interpretation and point of view are not particularly useful when the densities are considered uniform within each cell; the simpler and more direct approach of the present development then seems preferable. However, the foregoing observation does suggest that relaxation of the Dukowicz flux continuity condition will be of interest in connection with rezoning schemes based on higher order interpolations.

APPENDIX: Outline of the Logic for Sweeping Quadrilateral Mesh Lines

To illustrate the procedure for sweeping mesh lines and determining segment endpoints, it is sufficient to consider the case in which we are sweeping along a particular line of the new mesh, as shown in Fig. 2. The same procedure of course applies with the roles of the two meshes reversed. Suppose that work has just been completed on segment ax , so that we are located at the point a and are moving toward point b . We consider point a as endpoint 1 of the current new-mesh segment and we need to determine endpoint 2. As will be seen, this will also tell us which old-mesh cell the *next* new-mesh segment will lie in. Thus the fact that the current segment lies within cell (I, J) of the old mesh was determined as part of the work on the previous segment, and may be regarded as *given*.

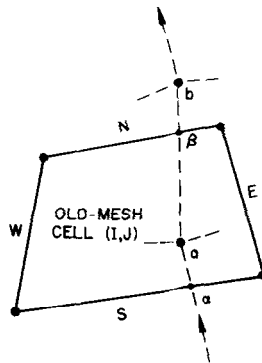


FIG. 2. Illustration of the procedure for sweeping along a new-mesh line (shown dashed) passing through old-mesh cell (I, J) .

The procedure begins by computing the points of intersection of the line segment ab with the line segments that form the N, S, E, and W sides of the old-mesh cell (I, J) . If there are no such intersection points then point b also lies within cell (I, J) of the old mesh, and point b itself is endpoint 2. In this case the next segment of the new-mesh line will clearly also lie in old-mesh cell (I, J) . If one or more such intersection points exist, the one nearest to a defines endpoint 2. In this case the next new-mesh segment will lie in the old-mesh cell adjacent to the side thus intersected by ab . For example, in Fig. 2 the intersection point β with side N defines endpoint 2 of the current new-mesh segment, and the *next* new-mesh segment will therefore lie in the old-mesh cell $(I, J+1)$ which shares side N with old-mesh cell (I, J) .

The procedure just outlined has the character of a marching method, in which one marches along a given line of one mesh determining each segment thereof, and the cell of the other mesh in which it lies, in a sequential manner. The topological structure of the mesh is used to determine which cell one enters upon leaving a given cell along a given line, and this eliminates the need to locate each new segment from scratch. Of course, in a computer program it is necessary to distinguish and deal with certain special cases that have not been considered in the above brief description, such as the intersection of a line of one mesh with a vertex of the other. It would be inappropriate to dwell on such cases here, as their treatment is straightforward and is largely a matter of personal programming preference.

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